Pythagorean Triples

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Historical Introduction

The investigation of Pythagorian triples has a very long history. For the first hundred years I refer to the famous book [DIC01]. Triangles of this type were given by Greek and Indian mathematicians. Arithmetically these are the solutions of the diophantine equation

$$x^2 + y^2 = z^2$$

in rational numbers. The general solution is given by the formulas

$$\begin{aligned} x &= l(m^2 - n^2) \\ y &= l \cdot 2mn \\ z &= l(m^2 + n^2) \qquad l\ (l \neq 0),\ m, n \text{ arbitrary}. \end{aligned}$$

These formulas are already contained in the works of Euclid and Brahmegupta. We also mention Bháscara, Pisano, Vieta,[1] Euler and Kronecker.

In early days the case $|x - y| = 1$ has been studied (example: $x = 3, y = 4, z = 5$). This leads to the Pell-Fermat equation

$$x^2 - 2z^2 = \pm 1.$$

It has been treated in the antiquity but is still of interest (In [RUN01] further references can be found. The paper [PAR01], unfortunately, was not accessible in the original version).

In this paper we will refer to further historic articles.

The intimate connection with the right-angled triangles and also with the unit circle given by

$$\xi = \cos\omega = \frac{1 - t^2}{1 + t^2}, \quad \eta = \sin\omega = \frac{2t}{1 + t^2}, \quad t = \frac{m}{n}$$

is well-known.

The first important progress, namely that $\frac{\omega}{\pi}$ is irrational, was made by Scherrer and Hadwiger. I refer to my article [HLA01]. This article contains new results which were presented in a lecture on Dec 9, 1977.

[1] F. Vieta, "Genesis triangolarum".

§1

Consider the Gaussian plain of numbers $\alpha = A + Bi$, where $i = \sqrt{-1}$ and A, B are integers. Here the norm $N(\alpha) = A^2 + B^2$ *satisfies* $N(\alpha) \geq 1$. The pair (A, B) is considered to be the point with the integer coordinates A and B. The set of these points forms a lattice with the unit sqare forming its fundamental area. This lattice is invariant under transformations T of the type

$$\begin{aligned} x' &= Ex + Fy \\ y' &= Gx + Hy, \end{aligned}$$

where E, F, G, H are integers with determinant 1. The corresponding inverse transformation is given by

$$\begin{aligned} x &= Hx' - Fy' \\ y &= -Gx' + Ey'. \end{aligned}$$

Together with α we consider its conjugate $\bar{\alpha} = A - iB$ and define

$$p(\alpha, \varepsilon) = \frac{\alpha}{\bar{\alpha}}\varepsilon = \frac{A + iB}{A - iB}\varepsilon, \tag{1}$$

where ε is one of the four different powers of i: $1, i, i^2 = -1, i^3 = -i$.

The set of all $p(\alpha, \varepsilon)$ forms a group with respect to multiplication with unit $E = p(1, 1)$ and inverse $p(\alpha, \varepsilon)^{-1} = p(\bar{\alpha}, \bar{\varepsilon})$. We call this group the pythagorian group P. All its members have norm 1. We call all $p(\alpha, \varepsilon)$ with $p(\alpha, 1)$ associated. If we choose one of them we write $p(\alpha)$. Note that among these numbers

$$\alpha = A + iB, \quad \alpha i = -B + Ai, \quad \alpha i^2 = -(A + Bi), \quad \alpha i^3 = B - Ai \tag{2}$$

resp. among the numbers

$$\bar{\alpha} = A - iB, \quad \bar{\alpha} i = B + Ai, \quad \bar{\alpha} i^2 = -A + iB, \quad \bar{\alpha} i^3 = -B - Ai$$

exactly one has positive real and imaginary part which we call the principal number of these four numbers. Here we may assume $A \geq B$, otherwise exchange A and B.

Multiplication of numerator and denominator in the fraction (1) with $\bar{\alpha}$, yields

$$p(\alpha) = X(\alpha) + iY(\alpha), \tag{3}$$

with

$$X(\alpha) = \frac{A^2 - B^2}{A^2 + B^2}, \quad Y(\alpha) = \frac{2AB}{A^2 + B^2}. \tag{4}$$

Putting

$$x = X(\alpha), \quad y = Y(\alpha), \quad z = p(\alpha) = x + iy, \tag{5}$$

we get

$$z\bar{z} = x^2 + y^2 = 1, \tag{6}$$

i.e. (x, y) is on the unit circle and the triple

$$(A^2 - B^2, 2AB, A^2 + B^2) = (a, b, c) \tag{7}$$

fulfills the diophantine equation

$$a^2 + b^2 = c^2. \tag{8}$$

We have

$$z = \frac{z+1}{\bar{z}+1}, \tag{9}$$

if $\bar{z} + 1 \neq 0$, hence $x + 1 \neq 0$.

Putting

$$t = \frac{y}{x+1} = \frac{B}{A}, \tag{10}$$

we get for $A \neq 0$

$$z = x + iy = \frac{1+it}{1-it} = \frac{(1+it)^2}{1+t^2}, \tag{11}$$

hence

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{2t}{1+t^2}, \tag{12}$$

the well known parameter representation of the unit circle, where the point $(-1, 0)$ is deleted.

The parameter representation in (4) is given in the homogeneous parameters (A, B) containing the point $(-1, 0)$ corresponding to $A = 0$, $B = 1$.

In (7) we suppose all multiples $(\lambda a, \lambda b, \lambda c)$, $\lambda \neq 0$ integer, to be contained.

In the author's paper [HLA01] we have gone the converse direction (here we refer to it briefly as "Pythagoräische Tripel I" or "PT I", while for the paper in front of you we write "PT II"). PT II can be read independently. Some results from PT I will be quoted and proofs will be sketched.

We want to describe the group P explicitly: If

$$\alpha_1 = A_1 + B_1 i, \quad \alpha_2 = A_2 + B_2 i,$$

then

$$\alpha_3 = \alpha_1\alpha_2 = A_3 + iB_3, \tag{13}$$

$$A_3 = A_1A_2 - B_1B_2, \quad B_3 = A_2B_1 + A_1B_2 \tag{14}$$

$$P(\alpha_3) = P(\alpha_1)P(\alpha_2).$$

Hence the triple (7) is given by

$$(a_3, b_3, c_3)$$

with

$$a_3 = A_3^2 - B_3^2, \quad b_3 = 2A_3B_3, \quad c_3 = A_3^2 + B_3^2 = (A_1^2 + B_1^2)(A_2^2 + B_2^2).$$

If $\alpha = A + Bi$, then (L integer)

$$\alpha^L = A_L + B_L i.$$

Let first L denote natural number, then

$$A_L + iB_L = (A + iB)^L = \sum_{k=0}^{L} \binom{L}{k} A^{L-k}(iB)^k. \tag{15}$$

We distinguish the case $k = 2r$ even (to get A_L) and the case $k = 2r + 1$ odd (to get B_L):

$$\begin{aligned} A_L &= \sum_r \binom{L}{2r}(-1)^r A^{L-2r} B^{2r} \\ B_L &= \sum_r \binom{L}{2r+1}(-1)^r A^{L-2r-1} B^{2r+1} \\ C_L &= (A^2 + B^2)^L, \end{aligned} \tag{16}$$

Here r takes all nonnegative integers with $r \leq \frac{1}{2}L$.

For negative $L = -m$, let $A_L = A_m$, $B_L = -B_m$, hence

$$\alpha^L = A_L + iB_L = A_m - iB_m = \bar{\alpha}^m,$$

In P we introduce a further operation,[2] which we denote by $\oplus$,

$$p(\alpha_0) = p(\alpha_1) \oplus p(\alpha_2).$$

For $\alpha_j = A_j + iB_j$, $j = 1, 2$, define

$$A_0 = A_1 A_2, \quad B_0 = B_1 B_2.$$

For all $j = 0, 1, 2$ we put

$$N_j = A_j^2 + B_j^2, \quad D_j = A_j^2 - B_j^2, \quad L_j = 2A_j B_j.$$

We get

$$p(\alpha_j) = X_j + iY_j = \frac{D_j + iL_j}{N_j}.$$

and $N_0 = |\alpha_0|^2 = A_0^2 + B_0^2$. An easy computation shows

$$N_0 = \frac{1}{2}(N_1 N_2 + D_1 D_2). \tag{o}$$

[2] which seems not to occur in the literature explicitly.

Furthermore we have

$$\alpha_0^2 = A_0^2 - B_0^2 + 2iA_0B_0 = D_0 + iL_0.$$

and

$$D_0 = \frac{1}{2}(D_1N_2 + D_2N_1), \quad L_0 = \frac{1}{2}L_1L_2, \tag{$\circ\circ$}$$

hence

$$p(\alpha_0) = \frac{\alpha_0}{\bar{\alpha}_0} = \frac{\alpha_0^2}{|\alpha_0|^2} = X_0 + iY_0.$$

Since

$$X_0 = \frac{D_0}{N_0}, \quad Y_0 = \frac{L_0}{N_0},$$

($\circ$) and ($\circ\circ$) yield

$$X_0 = \frac{x_1 + x_2}{1 + x_1x_2}, \quad Y_0 = \frac{y_1y_2}{1 + x_1x_2},$$

and therefore

$$\mathrm{Re}(p(\alpha_1) \oplus p(\alpha_2)) = \frac{\mathrm{Re}(p(\alpha_1) + p(\alpha_2))}{1 + \mathrm{Re}(p(\alpha_1))\,\mathrm{Re}(p(\alpha_2))}, \tag{17}$$

$$\mathrm{Im}(p(\alpha_1) \oplus p(\alpha_2)) = \frac{\mathrm{Im}(p(\alpha_1))\,\mathrm{Im}(p(\alpha_2))}{1 + \mathrm{Re}(p(\alpha_1))\,\mathrm{Re}(p(\alpha_2))}. \tag{18}$$

We want to bring out a consequence:

$$X_i^2 + Y_i^2 = 1,$$

holds for all i, since

$$|Y_i| = \sqrt{1 - X_i^2}\,.$$

Hence (18′) gets

$$\sqrt{1 - X_0^2} = \frac{\sqrt{1 - X_1^2}\sqrt{1 - X_2^2}}{1 + X_1X_2}, \tag{18$'$}$$

yielding

$$X_0^2 = 1 - (1 - X_0^2) = 1 - \frac{(1 - X_1^2)(1 - X_2^2)}{(1 + X_1X_2)^2}. \tag{18$''$}$$

We want to write (18′) in a different form: Define

$$\gamma(X_i) = \frac{1}{\sqrt{1 - x_i^2}}, \tag{19}$$

then (18″) can be written as

$$\gamma(X_0) = \gamma(X_1)\gamma(X_2)(1 + X_1X_2). \tag{20}$$

Now we involve geometric aspects. There is a unique angle φ_0 with $0 \le \varphi_0 < 2\pi$, such that for all φ of the type $\varphi_0 + 2\pi k$ (k takes all integer values, we write $\varphi \equiv \varphi_0 \pmod{2\pi}$)

$$p(\alpha) = e^{i\varphi} = \cos\varphi + i\sin\varphi,$$

hence

$$\cos\varphi = \frac{A^2 - B^2}{A^2 + B^2}, \quad \sin\varphi = \frac{2AB}{A^2 + B^2}.$$

More explicitly we write

$$\varphi = \operatorname{arc} p(\alpha).$$

Note

$$\operatorname{arc}(p(\alpha_1\alpha_2)) = \operatorname{arc} p(\alpha_1) + \operatorname{arc} p(\alpha_2) \pmod{2\pi} \tag{21}$$

$$\operatorname{arc}(p(\alpha)) = L \operatorname{arc} p(\alpha) \pmod{2\pi}. \tag{22}$$

Again we have

$$X = \frac{A^2 - B^2}{A^2 + B^2}, \qquad X = \frac{2AB}{A^2 + B^2}.$$

Considering the determinants

$$T_1 = \begin{vmatrix} X & 1 \\ Y & i \end{vmatrix} \qquad T_2 = \begin{vmatrix} X & 1 \\ Y & -i \end{vmatrix}$$

$$T_3 = \begin{vmatrix} 1 & 1 \\ 0 & i \end{vmatrix} \qquad T_4 = \begin{vmatrix} 1 & 1 \\ 0 & -i \end{vmatrix}$$

and their cross ratio we get

$$CR(T_1, T_3; T_2, T_4) = \frac{T_1}{T_3} : \frac{T_2}{T_4} = \left(\frac{A + iB}{A - iB}\right)^2,$$

hence (due to Laguerre)

$$\varphi_0 = \frac{1}{2i} \log CR(T_1, T_3; T_2, T_4).$$

This interpretation is often useful for applications of pythagorean triples.

We put

$$\varphi_0 = \pi\chi.$$

The Swiss Mathematicians Scherrer and Hadwiger have proved:
If $\alpha = A + iB$, $AB \neq 0$ and $A^2 \neq B^2$, then χ is irrational.[3]

[3][SCH01], [HAD01].

Proof: [4] Assume, by contradiction, that $\chi = \frac{m}{L}$ is rational. Then

$$\frac{\alpha}{\bar{\alpha}} = e^{\frac{i\pi m}{L}}$$

and

$$\left(\frac{\alpha}{\bar{\alpha}}\right)^L = e^{i\pi m} = \pm 1.$$

Since

$$\left(\frac{\alpha}{\bar{\alpha}}\right)^L = \frac{A_L + iB_L}{A_L - iB_L},$$

we get

$$A_L + iB_L = \pm(A_L - iB_L). \qquad (*)$$

We may assume that A_L and B_L have no common prime divisors. Let us make the assumption that

$$L \equiv 1 \pmod 2, \quad A \equiv 1 \pmod 2, \quad B \equiv 0 \pmod 2. \qquad (**)$$

Consider first the case of negative sign in (*), thus

$$A_L = 0.$$

Considering (16) modulo 2 leads to a contradiction, since with the exception of A (which is odd by assumption (**)) all terms have the even divisor B.

Suppose now that L is even:
Let 2^ρ the maximal power of 2 in L, i.e.

$$L = 2^\rho L_1,$$

L_1 odd. We put $\beta = \alpha^{2^\rho}$, then

$$\left(\frac{\alpha}{\bar{\alpha}}\right)^L = \left(\frac{\beta}{\bar{\beta}}\right)^{L_1}.$$

Let

$$e^{i\pi\chi_1} = \frac{\beta}{\bar{\beta}}.$$

Maintaining all other assumptions from (**), we see from the preceeding argument that χ_1 and, since $\chi_1 = 2^\rho\chi$, χ are irrational. Now cancel condition (**)!

Assume that A and B have no common prime divisors. It suffices to treat the case

$$A \equiv B \equiv 1 \pmod 2. \qquad (***)$$

Take α^2 instead of α. The corresponding triple is $(a_2, b_2, c_2) a_2 = A^2 - B^2,\ b_2 = 2AB$, $c_2 = (A^2 + B^2)^2$. (***) implies

$$A_2 \equiv 0 \pmod 4, \quad B_2 \equiv 0 \pmod 2,$$

[4] The proof presented here follows the book [MES01].

thus

$$A_3 = \frac{1}{2}A_2 \equiv 0 \pmod 2, \quad B_3 = \frac{1}{2}B_2 \equiv AB \equiv 1 \pmod 2.$$

Thus the triple (B_3, A_3, C_3) fulfills our conditions and the proof is complete. □

Consider the sequence $\omega = (2k\chi)$. Since 2χ is irrational, a result of the theory of uniform distribution (confer the textbooks [HLA02] and [KUI01]) tells us that the sequence is uniformly distributed modulo 1.

Now we compute the discrepancy D_N of the sequence. By Erdös-Turan's inequality we have

$$D_N \leq C\left(\frac{1}{M} + \sum_{|h|=1}^{M} \frac{1}{|h|}|W_N(h)|\right).$$

Here $W_N(h)$ is the Weyl sum

$$W_N(h) = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i h k \chi}.$$

We recall the easy computation. Note

$$|NW_N(h)| \leq \frac{1}{|1 - e^{2\pi i h \chi}|} = \frac{1}{|\sin^2 \pi h \chi|}$$

and

$$\sin \pi h \chi = \frac{B_h}{C^h},$$

with $C = A^2 + B^2$ ist.

B_h is an integer and, since χ is irrational, we have $B_h \neq 0$. Thus we get

$$|\sin \pi h \chi| \geq \frac{1}{C^h}$$

and we have shown that

$$|W_N(h)| \leq C^h.$$

The choice $M = [\frac{\log N}{\log C}] + 1$ leads to

$$D_N \leq \frac{20 \log C}{\log N}. \tag{23}$$

We will use this formula several times.

If f is integrable in the Riemann sense and with a period 1, then

$$\frac{1}{N}\sum_{k=1}^{N} f(k\chi) = \int_0^1 f(x)dx + \vartheta\, \sigma(D_N^{\frac{1}{3}}, f),$$

where $|\vartheta| \leq 1$ and $\sigma(\varepsilon, f)$ denotes the integrability module ε of f. If f has bounded variation $V(f)$, then we can give the more concrete estimation

$$\lambda_N(f) = \frac{1}{N}\sum_{k=1}^{N} f(kx) = \int_0^1 f(x)dx + \vartheta V(f) D_N.$$

If f has the form $G(\cos 2\pi x, \sin 2\pi x)$ (G integrable in E^2), then, provided that $G(\cos x, \sin x)$ has bounded variation $V(G)$, we get

$$\lambda_N(G) = \int_0^1 G(\cos 2\pi x, \sin 2\pi x)dx + \vartheta V(f) D_N,$$

where

$$\lambda_N(G) = \frac{1}{N}\sum G(a_{2L}, b_{2L})$$

with

$$a_{2L} = \frac{A_{2L}^2 - B_{2L}^L}{A_{2L}^2 + B_{2L}^L}, \quad b_{2L} = \frac{2A_{2L}B_{2L}}{A_{2L}^2 + B_{2L}^L}$$

and

$$A_{2L}^2 + B_{2L}^2 = (A^2 + B^2)^{2L}.$$

For differentiable G we have

$$V(G) \leq \operatorname*{Max}_{E^2} \sqrt{\left(\frac{\partial G}{\partial x}\right)^2 + \left(\frac{\partial G}{\partial y}\right)^2}.$$

§2

It is useful to transfer the uniformly distributed sequences considered above to the higher dimensional case. Consider a prime of the type $p = 4k + 1 = A^2 + B^2$ where the representation as a sum of two integer squares is unique up to the sign and order of A and B. Using complex numbers we have the representation

$$p = (A + iB)(A - iB) = \pi(p) \cdot \bar{\pi}(p),$$

where $\pi(p), \bar{\pi}(p)$ are different primes in the number field $Z(i)$. We have

$$\frac{\pi(p)}{\bar{\pi}(p)} = e^{i\pi\chi(p)}, \tag{1}$$

where χ is irrational and the sequence $(k\chi)$ is uniformly distributed mod 1.

If $p_1, p_2, \ldots, p_s$ are pairwise different primes of the type $4k + 1$ with corresponding $(\pi_1, \bar{\pi}_1), \ldots, (\pi_s, \bar{\pi}_s)$ and angles $\chi_1, \ldots, \chi_s$, then these angles are linearly independent in the sense of uniform distribution. The reason is that the prime decomposition in $Z(i)$ is unique. A consequence is that the sequence $(k\chi_1, \dots, k\chi_s)$ is uniformly distributed mod 1 in E^s. In the author's paper ([HLA03]) the proof of the estimation

$$D_N^s(p_1, \dots, p_s) \le 4^s C_s (\log P_s) \frac{(\log\log N)^s}{\log N} \tag{2}$$

for $P_s = p_1 \cdot \dots \cdot p_s$ can be found.

In

$$p_j = (A_j + iB_j)(A_j - iB_j) = A_j^2 + B_j^2 \tag{3}$$

the numbers A_j and B_j are given by (this representation is due to Jacobsthal)

$$A_j = -\frac{1}{2}\sum_{k=1}^{p_j}\left(\frac{k}{p_j}\right)\left(\frac{k^2+r_j}{p_j}\right) \tag{4}$$

$$B_j = -\frac{1}{2}\sum_{k=1}^{p_j}\left(\frac{k}{p_j}\right)\left(\frac{k^2+u_j}{p_j}\right), \tag{5}$$

where the r_j are the quadratic residues modulo p_j, and u_j the quadratic non-residues modulo p_j. For r_j one may independently take the residue $= -1$. For the non-residues one has to look among the numbers $1, \dots, 5\sqrt{p_j}$ (confer [VIN01]).

There are many results on the A_j and B_j, even recent investigations. For instance, the case that also A_j is a prime is treated in [FOU01].

As an example we give

$$p_1 = 5,\ \pi_1 = 1 + 2\sqrt{-1},\ p_2 = 13,\ \pi_2 = 3 + 2\sqrt{-1},\ p_3 = 17,\ \pi_3 = 4\sqrt{-1}.$$

§3 Applications

Applications to formulas of relativity theory. Consider the famous energy formula

$$E = \frac{m_0}{\sqrt{1-v^2}}.$$

Expanding of $(1 - v^2)^{-k}$ in terms of powers of v^2 and taking only two terms one gets

$$(1-v^2)^{-\frac{1}{2}} \sim 1 + \frac{1}{2}v^2,$$

and

$$E \sim m_0\left(1 + \frac{1}{2}v^2\right). \tag{6}$$

Taking for v the value

$$v_L = \frac{A_L^2 - B_L^2}{A_L^2 + B_L^2}, \tag{7}$$

The **theorem of additivity** §1 (17) is

$$\left(v_L \oplus v_m = \frac{v_L + v_m}{1 + v_L v_m}\right)$$

we get

$$E_L = \frac{1}{2} m_0 \left(\frac{A_L}{B_L} + \frac{B_L}{A_L}\right) \tag{8}$$

and for the impulse $p = \frac{m_0 v}{\sqrt{1-v^2}}$,

$$p_L = \frac{m_0 v_L}{\sqrt{1-v^2}} = \frac{1}{2}\left(\frac{A_L}{B_L} - \frac{B_L}{A_L}\right) m_0 . \tag{9}$$

We have

$$E_L^2 - p_L^2 = m_0^2 \,. \tag{10}$$

It is interesting to insert (2) in (1) to get

$$\hat{E}_L \sim m_0 \left(1 + \frac{1}{2} v_L^2\right) = m_0 \left(1 + \frac{1}{2}\left(\frac{A_L^2 - B_L^2}{A_L^2 + B_L^2}\right)^2\right).$$

The difference

$$E_v - \hat{E}_v$$

shows the error.

Now we consider the change of the mass

$$\Delta m_0 = 2 m_0 \left(\frac{1}{\sqrt{1-v^2}} - 1\right)$$

of two particles of the same mass coming together. For $v = v_L$ we get

$$\Delta_L m_0 = m_0 \frac{(|A| - |B|)^2}{|AB|} \,. \tag{11}$$

Computing for instance

$$\frac{1}{N} \sum_{L=1}^{N}{}^{*} 1$$

under the assumption that v_L lies in the interval $J : 0 < \alpha < \beta < 1$, we get from the theory of uniform distribution

$$\int_{\alpha}^{\beta} \frac{dv}{\sqrt{1-v^2}} + \text{error} = \arcsin\beta - \arcsin\alpha + \text{error}$$

and

$$\frac{1}{N}\sum_{L=1}^{N}{}^{*}(1-v_L^2)^{-1} = \operatorname{arctg}\gamma - \operatorname{arctg}\delta + \text{error},$$

where $\gamma = \arccos\beta,\ \delta = \arccos\alpha$.

The error terms can be estimated using the discrepancy.

Putting $v = v_L$ the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1-v^2}}, \qquad t' = \frac{vx + t}{\sqrt{1-v^2}} \tag{12}$$

gets to

$$\begin{aligned} x'_L &= \frac{1}{2}\left(\frac{A_L}{B_L} + \frac{B_L}{A_L}\right)x + \frac{1}{2}\left(\frac{A_L}{B_L} - \frac{B_L}{A_L}\right)t \\ t'_L &= \frac{1}{2}\left(\frac{A_L}{B_L} - \frac{B_L}{A_L}\right)x + \frac{1}{2}\left(\frac{A_L}{B_L} + \frac{B_L}{A_L}\right)t. \end{aligned} \tag{13}$$

Important points

$$x_L = c\frac{1}{2}\left(\frac{A_L}{B_L} + \frac{B_L}{A_L}\right), \quad t_L = c\frac{1}{2}\left(\frac{A_L}{B_L} - \frac{B_L}{A_L}\right)$$

lie on the curve $x^2 - t^2 = c^2$.

In the three dimensional case the Lorentz transformation is given by

$$\begin{aligned} \hat{x}_L &= \mathbf{x} - (xo)\left(1 - \frac{1}{\sqrt{1-v_L^2}}\right) + \frac{v_L t}{\sqrt{1-v_L^2}} \\ \hat{t}_L &= \frac{t - v_L(xo)}{\sqrt{1-v_L^2}}, \end{aligned}$$

where

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3), \quad \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \\ o &= (o_1, o_2, o_3) \qquad \text{unit vector} \\ (xo) &= x_1 o_1 + x_2 o_2 + x_3 o_3. \end{aligned}$$

The case $\tilde{v}_L = -\frac{1}{v_L}$ ($|\tilde{v}_L| > 1$) (Dirac electron) is also important. We do not go further into this direction.

The sequence of the pairs $(v_L, \tilde{v}_L)$ für $l = 1, 2, \ldots$ forms a countable model of a 'Dirac sea' (maybe this is an interpretation of a remark of Hilbert 1934 reported by the physicist Sauter).

A little application to the general relativity theory: Using the equivalence principle we get the Schwarzschild radii r_k:

$$r_k = \frac{2GM}{v_k^2} = 2GM\left(\frac{A_k^2 - B_k^2}{A_k^2 + B_k^2}\right)^2.$$

Application to the triangle: If a right-angled triangle has the legs x, y and the hypotenuse z then, by Pytharoras' theorem, $x^2 + y^2 = z^2$ and the triple $(A^2 - B^2, 2AB, A^2 + B^2)$ satisfies the equation. Frequently z and x are given and y has to be computed.

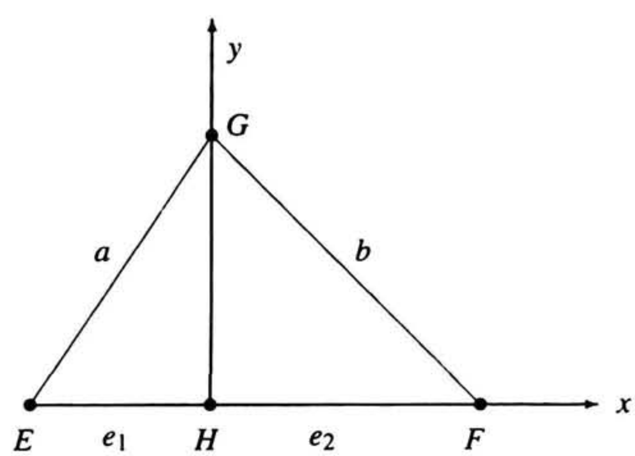
Starting with the relation $z^2 - y^2 = x^2$ we get the solution

$$z = \frac{1}{2}\left(\frac{A}{B} + \frac{B}{A}\right)x, \quad y = \frac{1}{2}\left(\frac{A}{B} - \frac{B}{A}\right)x.$$

We get triples of a type treated above.

The following idea is due to R. Lauffer:[5]

Consider an arbitrary triangle with vertices E, F, G arranged as in the picture.



Let the coordinates be given by $E = (-e_1, 0)$, $F = (e_2, 0)$, $G = (0, h)$, $H = (0, 0)$ and the lengths of the edges be a for EG, b for FG. The length of EH ist e_1, that of HF is e_2.

Let p_1, p_2 be primes with the corresponding Gaussian primes $\pi_1 = A_1 + iB_1, \pi_2 = A_2 + iB_2$. For

$$a = \frac{1}{2}\left(\frac{A_1}{B_1} + \frac{B_1}{A_1}\right)h, \quad b = \frac{1}{2}\left(\frac{A_2}{B_2} + \frac{B_2}{A_2}\right)h,$$

we have

$$e_1 = \frac{1}{2}\left(\frac{A_1}{B_1} - \frac{B_1}{A_1}\right)h, \quad e_2 = \frac{1}{2}\left(\frac{A_2}{B_2} - \frac{B_2}{A_2}\right)h.$$

[5] **Historical remark: Before Lauffer there were several mathematicians treating related problems: Brahmagupta, Euler, Gauß, Blichfeld and, mainly H. Schubert. We again mention Dickson. Unfortunately, Schubert's original papers were not accessible for me.**

The total length EF is

$$\frac{1}{2}\left(\frac{A_1}{B_1}+\frac{A_2}{B_2}+\left(\frac{B_1}{A_1}-\frac{B_2}{A_2}\right)\right)h.$$

We compute the angles α in E, β in F and γ in G we get

$$\cos\alpha = \frac{A_1^2 - B_1^2}{A_1^2 + B_1^2}, \qquad \sin\alpha = \frac{2A_1B_1}{A_1^2 + B_1^2}, \tag{14}$$

$$\cos\beta = \frac{A_2^2 - B_2^2}{A_2^2 + B_2^2}, \qquad \sin\beta = \frac{2A_2B_2}{A_2^2 + B_2^2}, \tag{15}$$

(the formula for $\sin\alpha$ is due to H. Schubert).

To compute γ we use

$$\cos\gamma = \cos(\pi(-(\alpha+\beta))) = \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

and

$$\sin\gamma = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$$

which follows from above. Thus the trigonometric functions of the angles are rational numbers. If h is a rational number, then the same holds for the edges.

There are applications of the pythagorean triples to different branches of mathematics:

rotations
3-dimensional pythagorean triples
quadratures
interpolation on the unit circle
rational points on spheres
line geometry
Clifford surfaces
spherical and non-euclidean geometry
infinite series
partial differential equations
'natural geometry'.

These applications are part of the content of a manuscript which is to be published later on.

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